2015-2016 MM2MS2 Exam Solutions

1.

(a)

For the thin walled cylinder, it can be assumed that $\sigma_r = 0$.

The axial stress due to internal pressure is:

$$\sigma_z = \frac{PR}{2t} = \frac{100 \times 200}{2 \times 10^{-3}} = 10\ MN/m^2$$

[2 marks]

Axial stress due to the axial load is:

$$\sigma_z = -\frac{F}{2\pi Rt} = -\frac{5 \times 10^3}{2\pi \times 200 \times 1 \times 10^{-6}} = -3.98\ MN/m^2$$

[2 marks]

So, the total axial stress is:

$$\sigma_z = 10 - 3.98 = \mathbf{6.02\ MN/m^2}$$

[1 mark]

And hoop stress due to internal pressure is:

$$\sigma_\theta = \frac{PR}{t} = \frac{100 \times 200}{10^{-3}} = \mathbf{20\ MN/m^2}$$

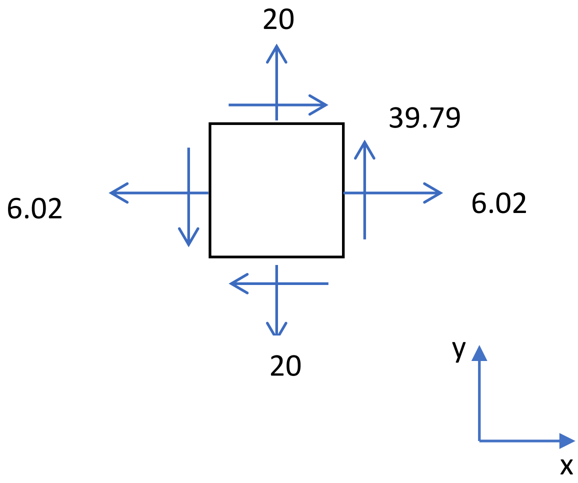
[2 marks]

Torsional stress is:

$$\tau = \frac{Tr}{J} = \frac{10 \times 200}{2\pi \times 0.2^3 \times 0.001} = \mathbf{39.79\ MN/m^2}$$

[2 marks]

Plane stress state of an element from the wall:



[3 marks]

(b)

The centre of the Mohr's circle:

$$C = \frac{(\sigma_x + \sigma_y)}{2} = \frac{20 + 6.02}{2} = 13.01\ \mathrm{MN/m^2}$$

[1 mark]

and the radius of the circle:

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{6.02 - 20}{2}\right)^2 + 39.79^2} = 40.40\ \mathrm{MN/m^2}$$

[2 marks]

$$\sigma_{1,2} = C \pm R = 13.01 \pm 40.40\ \mathrm{MN/m^2}$$

[1 mark]

Therefore, the principal stresses are:

$$\sigma_1 = \mathbf{53.41\ MN/m^2}$$

[2 marks]

$$\sigma_2 = \mathbf{-27.39\ MN/m^2}$$

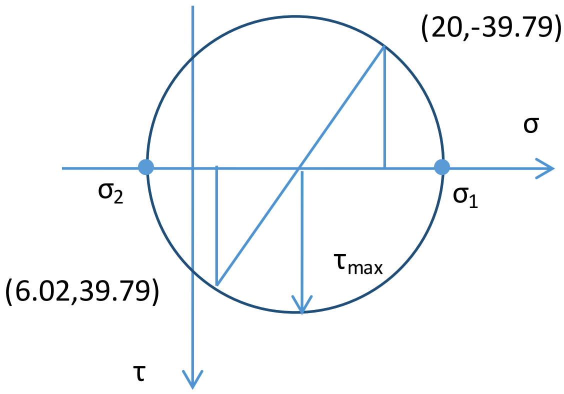
[2 marks]

The maximum shear stress is:

$$\tau_{max} = R = \mathbf{40.40\ MN/m^2}$$

[1 mark]

Mohr's circle



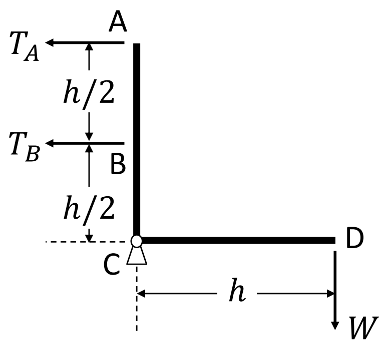
Not to scale

[4 marks]

2.

(a)

Free body diagram of the component:



Taking moments about position C:

$$W \times 2h = T_A \times 2h + T_B \times h$$

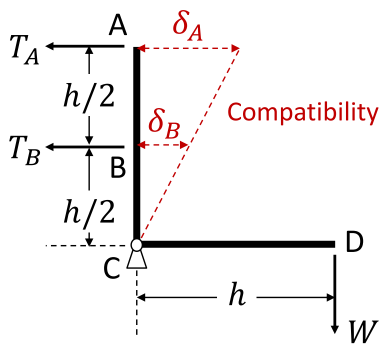
$$\therefore 2W = 2T_A + T_B \qquad (1)$$

[1 mark]

Also, from compatibility:

$$\delta_A = 2\delta_B$$

as shown in the following diagram:



$$\therefore \frac{T_A \times L}{EA} = 2\frac{T_B \times L}{EA}$$

$$\therefore T_A = 2T_B \qquad (2)$$

[1 mark]

Simultaneously solving equations (1) and (2) gives:

$$\boldsymbol{T_A = \frac{4}{5}W = 800\ \mathrm{N}}$$

$$\boldsymbol{T_B = \frac{2}{5}W = 400\ \mathrm{N}}$$

[3 marks]

(b)

$$\delta_A = \frac{T_A \times L}{EA} + \alpha\Delta TL$$

$$\delta_B = \frac{T_B \times L}{EA} + \alpha\Delta TL$$

where,

$$\delta_A = 2\delta_B$$

[4 marks]

Therefore,

$$\frac{T_A \times L}{EA} + \alpha\Delta TL = 2\frac{T_B \times L}{EA} + 2\alpha\Delta TL$$

$$\therefore T_A - 2T_B = EA\alpha\Delta T \qquad (3)$$

Also,

$$2W = 2T_A + T_B \qquad (4)$$

[2 marks]

Simultaneously solving equations (3) and (4) gives:

$$T_A = \frac{1}{5}(4W + EA\alpha\Delta T) = \frac{1}{5}(4000 + 240) = \mathbf{848\ N}$$

and,

$$T_B = 2(W - T_A) = 2(1000 - 848) = \mathbf{304\ N}$$

[4 marks]

(c)

Slacking:

$$T_B = 0$$

[4 marks]

Therefore, from equation (3):

$$T_A (= W) = EA\alpha\Delta T$$

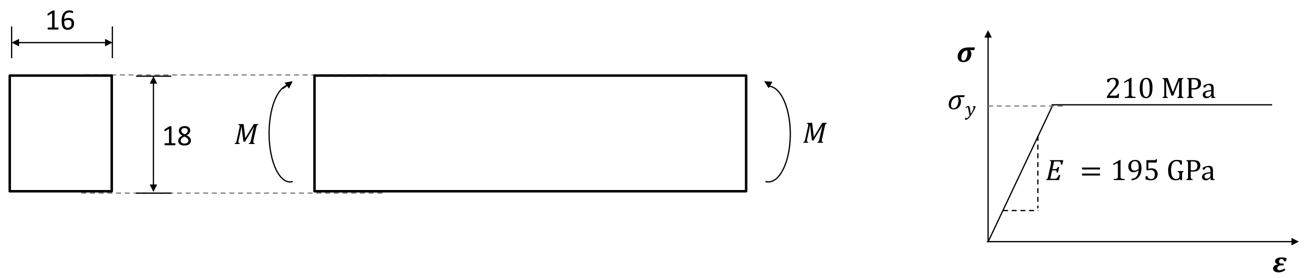
$$\therefore 1000 = 200{,}000 \times 12 \times 10^{-6} \times \Delta T$$

$$\therefore \Delta T = 416\ °\text{C}$$

$$T_{raise} = 416\ °\text{C} - 100\ °\text{C} = \mathbf{316\ °C}$$

[6 marks]

3.



$$I = \frac{bd^3}{12} = \frac{16 \times 18^3}{12} = 7{,}776 \text{ mm}^4$$

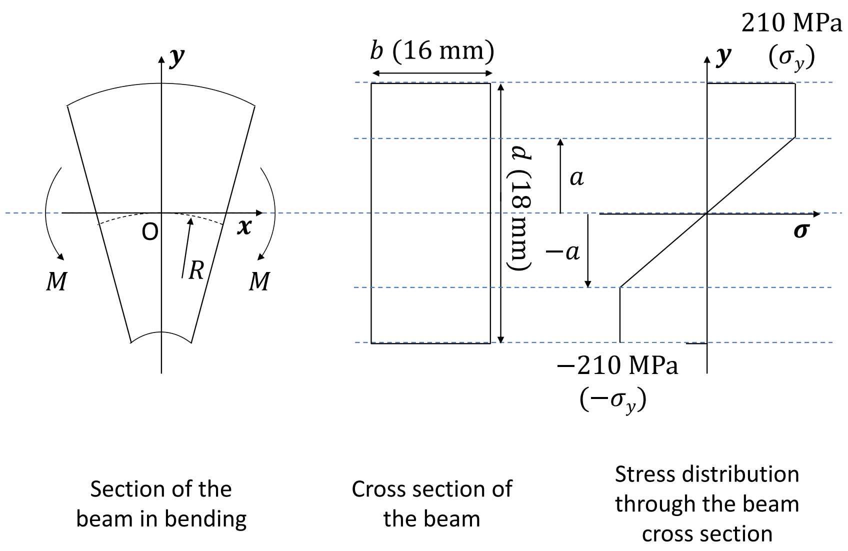
(a)

From beam bending equation:

$$\hat{\sigma} = \frac{M\hat{y}}{I} = \frac{205{,}000 \times 9}{7{,}776} = 237.27 \text{ MPa}$$

[1 mark]

Since $\sigma > \sigma_y$, plastic deformation will occur.



Variation of stress with y:

- For a $< y < 15, \sigma = 210$ MPa
- For $-$a $< y < a, \sigma = \frac{210}{a} y$ MPa
- For $-15 < y < -a, \sigma = -210$ MPa

[2 marks]

Moment equilibrium:

(Balance the moments due to stresses in the elastic and plastic regions with the applied moment)

$$M = \int_A y\sigma dA = \int_{-d/2}^{d/2} y\sigma b dy$$

[2 marks]

Due to the symmetry of the stress distribution, this can be rewritten as:

$$M = 2\int_0^{d/2} y\sigma b dy$$

[1 mark]

Substituting the elastic and plastic terms for σ into this:

$$M = 2\left\{\int_0^a y\frac{210}{a} y b dy + \int_a^{d/2} y(210) b dy\right\} = 2 \times 210b\left\{\int_0^a \frac{y^2}{a} dy + \int_a^{d/2} y dy\right\}$$

$$= 420b\left\{\left[\frac{y^3}{3a}\right]_0^a + \left[\frac{y^2}{2}\right]_0^{d/2}\right\} = 420b\left\{\left(\frac{a^3}{3a}\right) + \left(\frac{\left(d/2\right)^2}{2} - \frac{a^2}{2}\right)\right\}$$

$$\therefore M = 420b\left\{\frac{d^2}{8} - \frac{a^2}{6}\right\}$$

[2 marks]

Substituting the values of b, d and the applied moment, M, into this:

$$205{,}000 = 420 \times 16\left\{\frac{18^2}{8} - \frac{a^2}{6}\right\}$$

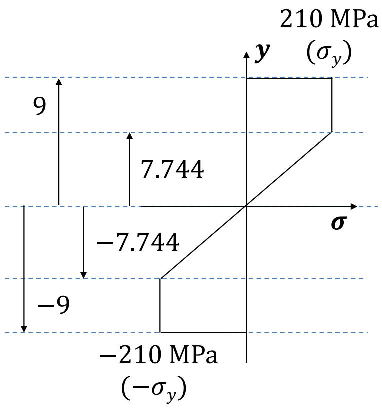
$$\therefore a^2 = 59.964$$

$$\therefore a = \pm\sqrt{59.964} = \pm 7.744 \text{ mm}$$

[2 marks]

Stress state in beam:

(Assuming the same material properties in compression as in tension)



[5 marks]

(b)

Assuming all unloading is elastic:

$$\frac{M}{I} = \frac{\sigma}{y}\left(= \frac{E}{R}\right)$$

$$\therefore \frac{\Delta M}{I} = \frac{\Delta \sigma}{y}$$

[2 marks]

Max change in stress ($\Delta\sigma$) will occur at $y = y_{max}$ $(= \pm 9\ \text{mm})$.

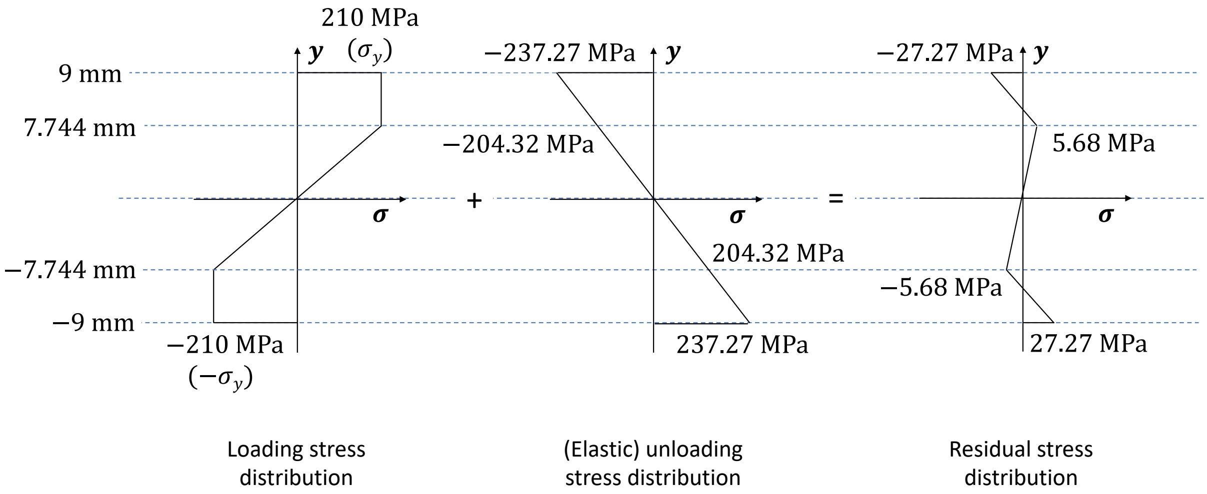
$$\therefore \Delta\sigma_{max}^{el} = \frac{\Delta M \times y_{max}}{I} = \frac{-M \times y_{max}}{I} = \frac{-205{,}000 \times \pm 9}{7{,}776} = \mp 237.27\ \text{MPa}$$

i.e.:

$$at\ y = 9\ \text{mm}, \therefore \Delta\sigma_{max}^{el} = -237.27\ \text{MPa}$$

$$and\ at\ y = -9\ \text{mm}, \therefore \Delta\sigma_{max}^{el} = 237.27\ \text{MPa}$$

[1 mark]



Interpolation of (elastic) unloading line:
At $y = 9$ mm, $\sigma = -237.27$ MPa

$$y = m\sigma + c$$
$$\therefore 9 = \mathrm{m} \times -237.27 + 0$$
$$\therefore \mathrm{m} = -0.0379$$

At $y = 7.744$ mm, $7.744 = -0.0379 \times \sigma$

$$\therefore \sigma = -204.32 \text{ MPa}$$

[2 marks]

Residual stress is well below yield (210 MPa), so reverse yielding does not occur. At $y = 7.744$ mm, no plastic deformation occurs during loading and unloading.

Compatibility:

Beam bending equation:

$$\frac{M}{I} = \frac{\sigma}{y} = \frac{E}{R}$$

$$\therefore \frac{y}{R} = \varepsilon \left(= \frac{\sigma}{E}\right)$$

$$\therefore R = \frac{y}{\varepsilon} \qquad (1)$$

[1 mark]

$\sigma - \varepsilon$ relationship:

At $y = a$ (outermost elastic point), the elastic relation, $\sigma = E\varepsilon$ is applicable. Therefore at $y = \pm 7.744$ $(= \pm a)$:

$$\pm\varepsilon_{residual} = \frac{\pm\sigma_{redidual}}{E} = \frac{\pm 5.68}{210{,}000} = 2.705 \times 10^{-5}$$

[1 mark]

Applying this to (1) gives:

$$R = \frac{y\ (at\ a)}{\varepsilon} = \frac{7.744}{2.705 \times 10^{-5}}$$

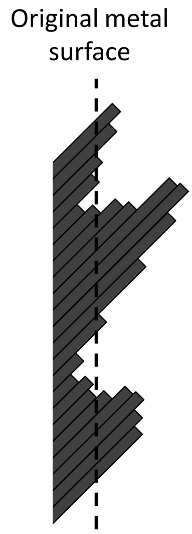
$$\therefore \boldsymbol{R} = \mathbf{286,284.66\ mm = 286.28\ m}$$

[3 marks]

4.

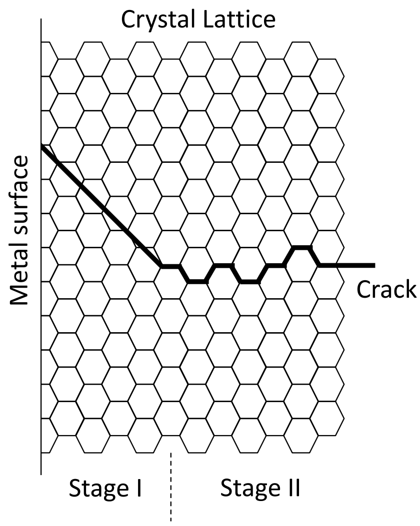
(a)

Crack Initiation and Stage I crack Growth: Pile up of dislocations causes slip bands, creating stress concentrations (features). This leads to shear stress controlled transgranular cracking. This occurs on the plane on maximum shear (45° to the loading plane) as shown below.



[3 marks]

Stage II Crack Growth: Once the crack has reached a critical length, the stress state around the crack tip changes and cracks will propagate due to the maximum tensile stress (90° to the loading direction). This phase is usually intergranular.

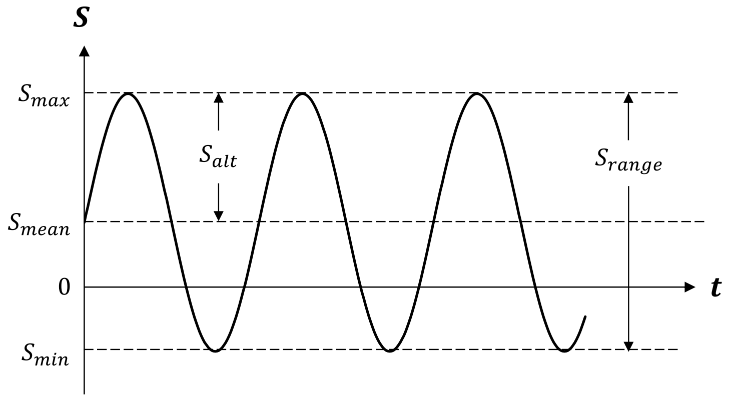


[4 marks]

Failure: Failure is at a critical crack length when the structure can no longer support the applied loads and it fails due to ductile tearing or cleavage (brittle) fracture.

[3 marks]

(b)



$$S_{max} = \text{Maximum Stress}$$

$$S_{min} = \text{Minimum Stress}$$

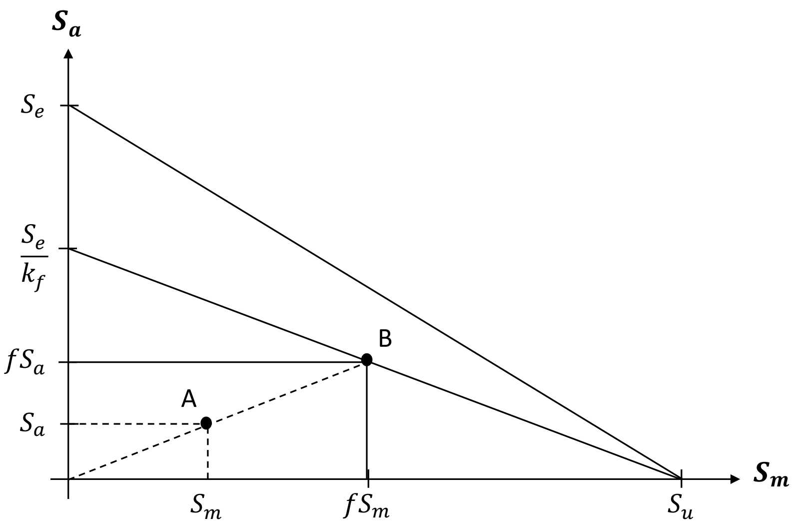
$$S_{mean} = \text{Mean Stress}$$

$$S_{range} = \text{Stress Range} = S_{max} - S_{min}$$

$$S_{alt} = \text{Alternating Stress} = \frac{S_{max} - S_{min}}{2}$$

[5 marks]

(c)



[4 marks]

From similar triangles:

$$\frac{S_e}{k_f S_u} = \frac{fS_a}{S_u - fS_m}$$

$$\therefore S_a = \frac{S_e(S_u - fS_m)}{fk_fS_u} = \frac{105(294 - 1.25 \times 45)}{1.25 \times 2 \times 294}$$

$$\therefore \boldsymbol{S_a = 33.96\ \mathrm{MPa}}$$

[6 marks]

5.

(a)

$$\begin{Bmatrix} \boldsymbol{F_1} \\ \boldsymbol{F_2} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{k_1} & \boldsymbol{-k_1} \\ \boldsymbol{-k_1} & \boldsymbol{k_1} \end{bmatrix} \begin{Bmatrix} \boldsymbol{u_1} \\ \boldsymbol{u_2} \end{Bmatrix}$$

[3 marks]

where,

$$k = AE/L$$

[2 marks]

(b)

The stiffness matrix of a truss element is:

$$[K_e] = \left(\frac{AE}{L}\right) \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta & -\cos^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta & -\cos\theta\sin\theta & -\sin^2\theta \\ -\cos^2\theta & -\cos\theta\sin\theta & \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & -\sin^2\theta & \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

Element 1, angle = 0°, $\cos\theta = 1$, $\sin\theta = 0$:

$$[K_{e1}] = (200 \times 10^6) \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[3 marks]

Element 2, angle = 45°, $\cos\theta = 0.7071$, $\sin\theta = 0.7071$:

$$[K_{e2}] = (200 \times 10^6) \begin{bmatrix} 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

[3 marks]

Overall stiffness matrix for structure

$$[\boldsymbol{K}] = (\boldsymbol{200 \times 10^6}) \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{-1} & \boldsymbol{0} & \boldsymbol{1.5} & \boldsymbol{0.5} & \boldsymbol{-0.5} & \boldsymbol{-0.5} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0.5} & \boldsymbol{0.5} & \boldsymbol{-0.5} & \boldsymbol{-0.5} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-0.5} & \boldsymbol{-0.5} & \boldsymbol{0.5} & \boldsymbol{0.5} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{-0.5} & \boldsymbol{-0.5} & \boldsymbol{0.5} & \boldsymbol{0.5} \end{bmatrix}$$

[4 marks]

(c)

Horizontal and vertical components of force at B:

$$F_{B_H}\,(F3) = 20\cos(-30) = 17.32 \text{ kN}$$

$$F_{B_V}(F4) = 20\sin(-30) = -10 \text{ kN}$$

[1 mark]

Overall equations:

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{bmatrix} = (200 \times 10^6) \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1.5 & 0.5 & -0.5 & -0.5 \\ 0 & 0 & 0.5 & 0.5 & -0.5 & -0.5 \\ 0 & 0 & -0.5 & -0.5 & 0.5 & 0.5 \\ 0 & 0 & -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}$$

Applying BCs, $u_1 = u_2 = u_5 = u_6 = 0$, reduces the problem to:

$$\begin{bmatrix} F_3 \\ F_4 \end{bmatrix} = (200 \times 10^6) \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

Applying forces:

$$\begin{bmatrix} 17320 \\ -10000 \end{bmatrix} = (200 \times 10^6) \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} u_3 \\ u_4 \end{bmatrix}$$

[1 mark]

Therefore:

$$17320 = 200 \times 10^6 \times (1.5u_3 + 0.5u_4)$$

$$8.66 \times 10^{-5} = 1.5u_3 + 0.5u_4 \qquad (1)$$

$$8.55 \times 10^{-5} - 0.5u_4 = 1.5u_3$$

$$5.7733 \times 10^{-5} - \left(\frac{1}{3}\right) u_4 = u_3$$

$$-\frac{10000}{200} \times 10^6 = 0.5u_3 + 0.5u_4$$

Subs in for u_3:

$$-\frac{10000}{200} \times 10^6 = 2.88665 \times 10^{-5} + \left(\frac{2}{6}\right) u_4$$

$$\left(\frac{2}{6}\right)u_4 = -5\times10^{-5} - 2.886655\times10^{-5}$$

$$\therefore \boldsymbol{u_4} = \mathbf{-2.36\times10^{-4}\ m = -0.236\ mm}$$

[1 mark]

Subs into (1):

$$u_3 = \frac{8.66\times10^{-5} - 0.5\times(-2.36\times10^{-4})}{1.5}$$

$$\therefore \boldsymbol{u_3} = \mathbf{1.36\times10^{-4}\ m = 0.136\ mm}$$

[1 mark]

(d)

From matrix equation:

$$F_1 = -200\times10^6 u_3 = \mathbf{-27280\ N}$$

[2 marks]

$$\boldsymbol{F_2} = \mathbf{0\ N}$$

[1 mark]

$$F_5 = 200\times10^{-6}\times(-0.5u_3 - 0.5u_4) = \mathbf{10000\ N}$$

[2 marks]

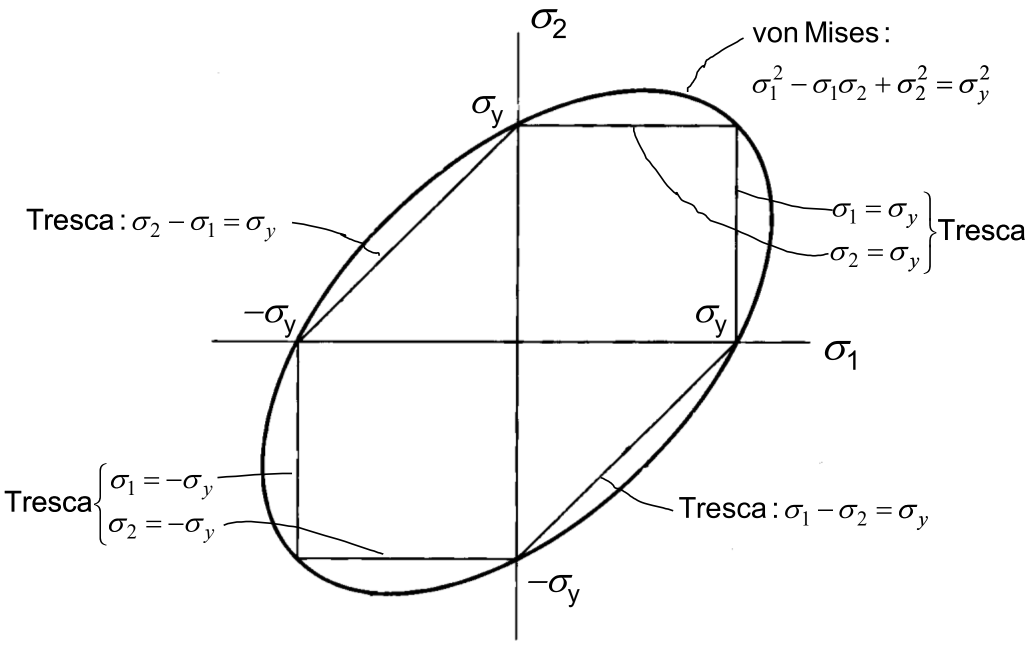
$$\boldsymbol{F_6} = \mathbf{10000\ N}$$

(equation identical to that for F_5)

[1 mark]

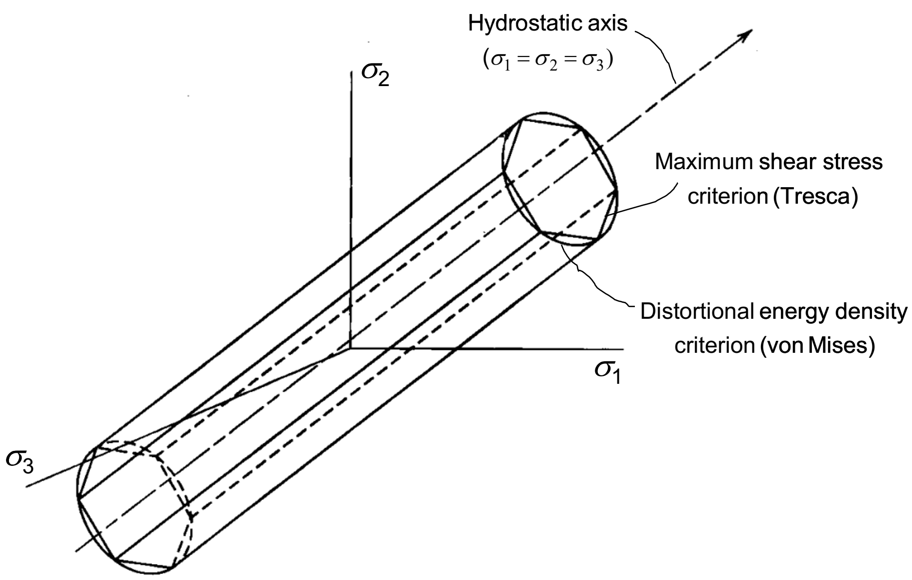
6.

(a)



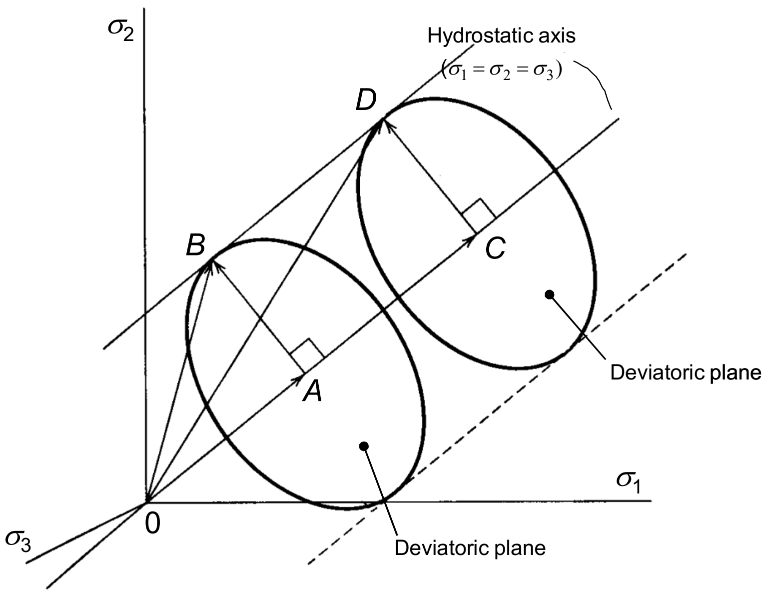
[3 marks]

(b)



[3 marks]

(c)



[3 marks]

(d)

$$\tau = \frac{Tr}{J}$$

$$J = \frac{\pi r^4}{2}$$

$$\sigma_b = \frac{My}{I}$$

$$I = \frac{\pi r^4}{4}$$

Where:

$$y = r$$

[2 marks]

Therefore:

$$\tau = \frac{16000}{\pi r^3}$$

$$\sigma_b = \frac{16000}{\pi r^3}$$

Mohr's Circle:

$$C = \frac{\sigma_b}{2} = \frac{8000}{\pi r^3}$$

$$R = \sqrt{\left(\frac{\sigma_b}{2}\right)^2 + \tau_{xy}^2}$$

For Tresca $\tau_{max} = R = 100$ MPa at yield (including S.F)

[2 marks]

Therefore:

$$(100 \times 10^6)^2 = \left(\frac{8000}{\pi r^3}\right)^2 + \left(\frac{16000}{\pi r^3}\right)^2$$

$$\therefore 100 \times 10^6 = \frac{17888}{\pi r^3}$$

$$\therefore r = \sqrt[3]{\frac{17888}{100 \times 10^6 \pi}} = \mathbf{0.0385\ m = 38.5\ mm}$$

[4 marks]

For von Mises

$$\sigma_y^2 = \sigma_1^2 - \sigma_1\sigma_2 + \sigma_2^2$$

[2 marks]

at yield (for a plane stress case), which can be interpreted as:

$$\sigma_y^2 = (C + R)^2 - (C + R)(C - R) + (C - R)^2$$

$$\therefore \sigma_y^2 = C^2 + 3R^2$$

yield stress is 200 MPa (including S.F),

[2 marks]

Therefore:

$$(200 \times 10^6)^2 = \left(\frac{8000}{\pi r^3}\right)^2 + 3 \times \left(\frac{17888}{\pi r^3}\right)^2$$

$$\therefore 200 \times 10^6 = \frac{31999}{\pi r^3}$$

$$\therefore r = \sqrt[3]{\frac{31999}{200 \times 10^6 \pi}} = \mathbf{0.03707\ m = 37.07\ mm}$$

[4 marks]

Alternative notation:

$$\sigma_1 = C + R = \frac{8000}{\pi r^3} + \frac{17888}{\pi r^3} = \frac{25888}{\pi r^3}$$

$$\sigma_2 = C - R = \frac{8000}{\pi r^3} - \frac{17888}{\pi r^3} = -\frac{9888}{\pi r^3}$$

$$\sigma_y^2 = \left(\frac{25888}{\pi r^3}\right)^2 - \left(\frac{25888}{\pi r^3}\right)\left(\frac{-9888}{\pi r^3}\right) + \left(\frac{-9888}{\pi r^3}\right)^2$$

$$\therefore \sigma_y^2 = \frac{1.024 \times 10^9}{\pi r^6}$$

$$200 \times 10^6 = \frac{31999}{\pi r^3}$$

$$\therefore r = \sqrt[3]{\frac{31999}{200 \times 10^6 \pi}} = \mathbf{0.03707\ m = 37.07\ mm}$$